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Homology and K-theory methods for classes of branes wrapping nontrivial cycles

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Abstract

We apply some methods of homology and K-theory to special classes of branes wrapping homologically nontrivial cycles. We treat the classification of four-geometries in terms of compact stabilizers (by analogy with Thurston's classification of three-geometries) and derive the K-amenability of Lie groups associated with locally symmetric spaces listed in this case. More complicated examples of T-duality and topology change from fluxes are also considered. We analyse D-branes and fluxes in type II string theory on $\mathbb{C}P^3 \times \Sigma_g \times \mathbb{T}^2$ with torsion H-flux and demonstrate in detail the conjectured T-duality to $\mathbb{R}P^7 \times X^3$ with no flux. In the simple case of $X^3 = \mathbb{T}^3$, T-dualizing the circles reduces to the duality between $\mathbb{C}P^3 \times \mathbb{T}^2 \times \mathbb{T}^2$ with H-flux and $\mathbb{R}P^7 \times \mathbb{T}^3$ with no flux.

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1. Introduction

The problem of classifying geometries is one of the main problems in complex analysis and in mathematics as a whole, and plays a fundamental role in physical models. Every one-dimensional manifold is either S^1 (closed, i.e. compact with empty boundary) or $\mathbb{R}$ (open), with a unique topological, piecewise linear and smooth structure and orientation. All curves of genus zero can be uniformized by rational functions, all those of genus one can be uniformized by elliptic functions, and all those of genus more than one can be uniformized by meromorphic functions, defined on proper open subsets of $\mathbb{C}$. This result, due to Klein, Poincaré and Koebe, is one of the deepest achievements in mathematics. A complete solution of the uniformization problem has not yet been obtained, except for the one-dimensional complex case. However, there were advances in this problem, which have been essential to understand the foundations of topological methods, covering spaces, existence theorems for partial differential equations, existence and distortion theorems for conformal mappings, etc.

In accordance with the Klein–Poincaré uniformization theorem, each Riemann surface can be represented (within a conformal equivalence) in the form Σ/Γ, where Σ is one of the three canonical regions, namely, the extended plane $\bar{\mathbb{C}}$ (the sphere $\mathbb{S}^2$), the plane $\mathbb{C}$ ($\mathbb{R}^2$), or the disc, and Γ is a discrete group of Möbius automorphisms of Σ acting freely there. Riemann surfaces with such coverings are elliptic, parabolic and hyperbolic type, respectively. This theorem admits a generalization also to surfaces with branching. With the help of the standard uniformization and decomposition theorems [1], one can construct and describe all the uniformizations of Riemann surfaces by Kleinian groups. Furthermore, by using the quasiconformal mappings, one can obtain an uniformization theorem of more general character[1]. Namely, it is possible to prove that several surfaces can be uniformized simultaneously. For any closed orientable two-dimensional manifold Σ_Γ the following result holds: every conformal structure on Σ_Γ is represented by a constant curvature geometry. The only simply connected manifolds with constant curvature are $\mathbb{S}^2$ or $\mathbb{R}^2$ or $\mathbb{H}^2$ and Σ_Γ can be represented as Σ/Γ, where Γ is a group of isometries.

An important progress in the three-dimensional case has been made by Thurston [2]. To analyse this case, we should consider the eight-geometric structure in the classification of three-manifolds introduced by Thurston. We also note the recent asserts for Ricci flow on the class of three-manifolds. The Ricci flow with surgery was considered in [3]. A canonical Ricci flow defined on the largest possible subset of spacetime has been constructed in [4, 5].

In this paper, we will analyse in detail the K-theory groups which are the more appropriate arena to classify the D-branes wrapping topologically nontrivial manifolds. We contemplate various three- and four-geometries giving the relevant meaning of low-dimensional brane cycles and describing some new results on K-amenability in the list of four-geometries. We concentrate our analysis on the application of the homological and K-theory methods to branes and Ramond–Ramond (RR) flux in type II string theory. The RR fields are typically supported on D-branes and take values in the appropriate K-theory groups [6]. These facts have prompted intensive investigations in both the mathematical and physical literature into the properties and definitions of various K-theory groups.

This paper is outlined as follows. The case of two-geometries is shortly discussed in section 2.1 and then in section 2.2 we present the Thurston's list of three-geometries. This list has been organized in terms of the compact stabilizers Γ_x of $x \in X$ isomorphic to $\mathbf{SO}(3)$, $\mathbf{SO}(2)$ or trivial group $\{1\}$. The analogue list for four-geometries and the corresponding stabilizer-subgroups of the kind $\mathbf{SO}(4)$, $\mathbf{U}(2)$, $\mathbf{SO}(2) \times \mathbf{SO}(2)$, $\mathbf{SO}(3)$, $\mathbf{SO}(2)$, $\mathbf{S}^1$ and $\{1\}$ are analysed in section 2.3.

The required mathematical tools considered in the paper are exposed in section 3. We also refer the reader to the appendix where the necessary material on Eilenberg–MacLane spaces is presented. The Kasparov's KK-pairings and the concept of K-amenable groups are considered in sections 3.1 and 3.2, respectively.

The different aspects of K-group theory are formulated in section 4. Twisted crossed product of C^*-algebras and twisted K-groups are considered in section 4.1.

Recall that some general theorems about the K-groups of C^*-algebras have been obtained in [7–9], while perfect analysis of Baum–Connes type conjectures on the K-theories of twisted C^*-algebras was considered in [10, 11]. In the special case of general statements, $K_*(C^*(\mathbb{Z}^n, \sigma)) \cong K_*(C^*(\mathbb{Z}^n)) \cong K^*(\mathbb{T}^n)$ for any multiplier (i.e. group two-cocycle) σ on $\mathbb{Z}^n$. The twisted group C^*-algebras $C^*(\mathbb{Z}^n, \sigma)$ have been called *noncommutative tori*.

[1] This fact is related to Techmüller spaces.

In [12, 13], this calculation was generalized for K-groups of the twisted group C^*-algebras of uniform lattices in solvable groups. Namely, if Γ is a uniform lattice in a solvable Lie group G, then

$$K_*(C^*(\Gamma, \sigma)) \cong K^{*+\dim G}(\Gamma\backslash G, \delta(B_\sigma)),$$

where σ is any multiplier on Γ, $K^*(\Gamma\backslash G, \delta(B_\sigma))$ denotes the twisted K-theory (see for detail [14]) of a continuous trace C^*-algebra B_σ with spectrum $\Gamma\backslash G$, and $\delta(B_\sigma) \in H^3(\Gamma\backslash G, \mathbb{Z})$ denotes the Dixmier–Douady invariant of B_σ. This result has been proved by using the Packer–Raeburn stabilization trick [12] and the Thom isomorphism theorem for the K-theory of C^*-algebras. Then in [15] the main theorem of [12, 13] has been extended to the case when Γ is a lattice in a K-amenable Lie group G [16]. For such G and Γ,

$$K_*(C^*(\Gamma, \sigma)) \cong K_*(C_r^*(\Gamma, \sigma)), \qquad K_*(C^*(\Gamma, \sigma)) \cong K^{*+\dim(G/K)}(\Gamma\backslash G/K, \delta(B_\sigma)),$$

where K is a maximal compact subgroup of G, σ is any multiplier on Γ, $K^*(\Gamma\backslash G/K, \delta(B_\sigma))$ is the twisted K-theory of a continuous trace C^*-algebra B_σ with spectrum $\Gamma\backslash G/K$.

For the three-dimensional case, these results are described in section 4.2. Then, in section 4.3 we prove the K-amenability for a set of Lie groups associated with locally symmetric spaces listed in the four-dimensional case. We formulate the following statement in the form of conjecture: if $K(\Gamma, 1)$ is a connected, compact, four-dimensional manifold which is an Eilenberg–MacLane space with fundamental group Γ, then for any multiplier $\sigma \in H^2(\Gamma, U(1))$ on Γ

$$K_j(C_r^*(\Gamma, \sigma)) \cong K_j(C_r^*(\Gamma)) \cong K^{j+1}(K(\Gamma, 1)), \qquad j = 0, 1.$$

The properties of the T-duality for principal higher rank torus bundles with H-flux have been studied in [17]. This construction of duality can be argued from the so-called generalized Gysin sequence. T-duality for torus bundles comes with the isomorphism of twisted cohomology and twisted cyclic homology, it was analysed via noncommutative topology in [18, 19]. It has been argued [20, 21] that for type II string theory a general principal torus bundle with general H-flux is a bundle of noncommutative, non-associative tori. Useful theorems to perform the T-duality computations for circle bundles with H-fluxes are given in [22, 23]. In section 5, we discuss some more complicated examples of T-duality for circle bundles and topology change from fluxes, taking into account the methods of homology and K-theory. We apply to D-branes classification and fluxes in type II string theory on $\mathbb{C}P^3 \times \Sigma_g \times \mathbb{T}^2$ with torsion H-flux (a similar model has been considered in [21, 24]) and demonstrate in detail the conjectured T-duality to $\mathbb{R}P^7 \times X^3$ with no flux. In the simple case of $X^3 = \mathbb{T}^3$, T-dualizing the circles reduces to duality between $\mathbb{C}P^3 \times \mathbb{T}^2 \times \mathbb{T}^2$ with H-flux and $\mathbb{R}P^7 \times \mathbb{T}^3$ with no flux.

2. Low-dimensional cycles

2.1. Two-geometries

Two-manifolds are complete classified: their piecewise linear and smooth structures are unique and depend only on the homeomorphic type. The homeomorphism type is determined by their fundamental group. If X is a closed orientable two-manifold, then it is either $\mathbb{S}^2$, $\mathbb{T}^2 = S^1 \times S^1$ or an g-fold connected sum $(\mathbb{T}^2)^{\# g}$.[2] If two-manifold X is not orientable, then it is the real projective plane $\mathbb{R}P^2$ or g-fold connected sum thereof, $(\mathbb{R}P^2)^{\# g}$. Surfaces of $g \geqslant 2$ are covered by the genus-two double-torus, which is hyperbolic, since it is the quotient of an

[2] The integer g is called the *genus* of the surface X and determines it and its fundamental group up to homeomorphism; we assume that $g \geqslant 1$.

Table 1. Two-space integral homologies.

Homologies	$n = 0$	$n = 1$	$n = 2$
$H_n(X_{\text{oriented}})$	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 2g}$	$\mathbb{Z}$
$H_n(X_{\text{non-oriented}})$	$\mathbb{Z}$	$\mathbb{Z}^{\oplus g-1} \oplus \mathbb{Z}/2$	0

octagon in hyperbolic disc. The integral homologies of surfaces of genus g are presented in table 1. If X is the orientable surface then $X = \mathbb{S}^2$ for $g = 0$, and $X = \mathbb{T}^2$ for $g = 1$. For non-orientable surface X we have $X = \mathbb{R}P^2$ and $X =$ the Klein bottle for $g = 1$ and 2, respectively. For orientable and non-orientable cases, the Euler characteristic $\chi(X) = \sum_\ell (-1)^\ell \, \mathrm{rk} H_\ell(X)$ satisfies $\chi(X) = 2 - 2g$ and $\chi(X) = 2 - g$, respectively.

2.2. Three-geometries

Following the presentation of [2, 25], by a geometry or a geometric structure we mean a pair (X, Γ) that is a manifold X and a group Γ acting transitively on X with compact point stabilizers (following [2] we also propose that the interior of every compact three-manifold has a canonical decomposition into pieces which have geometric structure). Two geometries (X, Γ) and (X', Γ') are equivalent if there is a diffeomorphism of X with X' which throws the action of Γ onto the action of Γ'. In particular, Γ and Γ' must be isomorphic. Assume that

- The manifold X is simply connected. Otherwise, it will be sufficient to consider a natural geometry $(\widetilde{X}, \widetilde{\Gamma})$, $\widetilde{X}$ being the universal covering of X and $\widetilde{\Gamma}$ denoting the group of all diffeomorphisms of $\widetilde{X}$ which are lifts of elements of Γ.
- The geometry admits a compact quotient. In other words, there exists a subgroup $\widehat{\Gamma}$ of Γ which acts on X as a covering group and has compact quotient.
- The group Γ is maximal. Otherwise, if $\Gamma \subset \Gamma'$ then any geometry (X, Γ) would be the geometry (X, Γ') at the same time.

Due to Thurston's conjecture there are eight model spaces in three dimensions:

$$X = G/K = \left\{ \begin{matrix} \mathbb{R}^3(\text{Euclidean space}), \mathbb{S}^3(\text{spherical space}), \ \mathbb{H}^3(\text{hyperbolic space}), \\ \mathbb{H}^2 \times \mathbb{R}, \mathbb{S}^2 \times \mathbb{R}, \widetilde{\mathbf{SL}(2, \mathbb{R})}, \mathbb{N}il^3, \mathbb{S}ol^3 \end{matrix} \right\}.$$

Remark 2.1. This conjecture follows from considering the identity component of the isotropy group Γ_x of X through a point x. Γ_x is a compact, connected Lie group, and there are three cases: $\Gamma_x = \mathbf{SO}(3), \mathbf{SO}(2)$ and $\{1\}$.

$\mathscr{R}1$. $\Gamma_x = \mathbf{SO}(3)$. In this case, the space X is a space of constant curvature: $\mathbb{R}^3, \mathbb{S}^3$ (modelled on $\mathbb{R}^3$) or $\mathbb{H}^3$ (which can be modelled on the half-space $\mathbb{R}^2 \times \mathbb{R}^+$).

$\mathscr{R}2$. $\Gamma_x = \mathbf{SO}(2)$. In this case, there is a one-dimensional subspace of $T_x X$ that invariant under Γ_x, which has a complementary plane field $\mathscr{P}_x$. If the plane field $\mathscr{P}_x$ is integrable, then X is a product $\mathbb{R} \times \mathbb{S}^2$ or $\mathbb{R} \times \mathbb{H}^2$. If the plane field $\mathscr{P}_x$ is non-integrable, then X is a nontrivial fibre bundle with fibre S^1: $S^1 \hookrightarrow X \twoheadrightarrow \Sigma_{g\geqslant 2}$ ($\widetilde{\mathbf{SL}(2, \mathbb{R})}$-geometry), Σ_g stands for a surface of genus g, $S^1 \hookrightarrow X \twoheadrightarrow \mathbb{T}^2$ ($\mathbb{N}il^3$-geometry) or $S^1 \hookrightarrow X \twoheadrightarrow \mathbb{S}^2$ ($\mathbb{S}^3$-geometry).

$\mathscr{R}3$. $\Gamma_x = \{1\}$. In this case, we have three-dimensional Lie groups: $\widetilde{\mathbf{SL}(2, \mathbb{R})}$, $\mathbb{N}il^3$ and $\mathbb{S}ol^3$.

Table 2. List of four-geometries.

Stabilizer-subgroup Γ_x	Space X
$\mathbf{SO}(4)$	$\mathbb{S}^4, \mathbb{R}^4, \mathbb{H}^4$
$\mathbf{U}(2)$	$\mathbb{C}P^2, \mathbb{CH}^2$
$\mathbf{SO}(2) \times \mathbf{SO}(2)$	$\mathbb{S}^2 \times \mathbb{R}^2, \mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2 \times \mathbb{H}^2, \mathbb{H}^2 \times \mathbb{R}^2, \mathbb{H}^2 \times \mathbb{H}^2$
$\mathbf{SO}(3)$	$\mathbb{S}^3 \times \mathbb{R}, \mathbb{H}^3 \times \mathbb{R}$
$\mathbf{SO}(2)$	$\mathbb{N}il^3 \times \mathbb{R}, \widetilde{\mathbf{PSL}}(2, \mathbb{R}) \times \mathbb{R}, \mathbb{S}ol^4$
$\mathbf{S}^1$	F^4
Trivial	$\mathbb{N}il^4, \mathbb{S}ol^4_{m,n}$ (including $\mathbb{S}ol^3 \times \mathbb{R}$), $\mathbb{S}ol^4_1$

The first five geometries are familiar objects, so we briefly discuss the last three ones. The group $\widetilde{\mathbf{SL}(2, \mathbb{R})}$ is the universal covering of $\mathbf{SL}(2, \mathbb{R})$, the three-dimensional Lie group of all 2×2 real matrices with unit determinant. The geometry of $\mathbb{N}il$ is the three-dimensional Lie group of all 3×3 real upper triangular matrices with ordinary matrix multiplication. It is also known as the nilpotent Heisenberg group. The geometry of $\mathbb{S}ol$ is the three-dimensional (solvable) group. Many three-manifolds are hyperbolic (according to a famous theorem by Thurston [2]). For example, the complement of a knot in $\mathbb{S}^3$ admits an hyperbolic structure unless it is a torus or satellite knot. Moreover, after the Mostow rigidity theorem [26], any geometric invariant of an hyperbolic three-manifold is a topological invariant. If a three-manifold X admits a geometric structure, then the universal cover $\widetilde{X}$ with the induced metric is isometric to one of the eight geometries above. X can admit more than one geometric structures, but if X is closed and admits a geometric structure then the geometric structure is unique: $\widetilde{X}$ is isomorphic to one and only one of the above geometries (for more detail, see [25]). The Euclidean space, being flat, does not lead to a new background in supergravity, i.e. the topological twist is trivial. Our special interest to hyperbolic spaces: it has been shown that for supergravity descriptions of branes wrapping three-dimensional cycles implies that cycles have to be a constant scalar curvature. The solution including $\mathbb{H}^3$ and its quotients by subgroup of isometry group $\mathbf{PSL}(2, \mathbb{C}) \equiv \mathbf{SL}(2, \mathbb{C})/\{\pm 1\}$ the reader can found in [27, 28].

2.3. *Four-geometries*

The list of Thurston three-geometries has been organized in terms of the compact stabilizers Γ_x of $x \in X$ isomorphic to $\mathbf{SO}(3), \mathbf{SO}(2)$ or trivial group $\{1\}$. The analogue list of four-geometries also can be organized (using only connected groups of isometries) as in table 2. Here we have the four irreducible four-dimensional Riemannian symmetric spaces: sphere $\mathbb{S}^4$, hyperbolic space $\mathbb{H}^4$, complex projective space $\mathbb{C}P^2$ and complex hyperbolic space $\mathbb{CH}^2$ (which we may identify with the open unit ball in $\mathbb{C}^2$ with an appropriate metric). The other cases are more specific and only for the sake of completeness we shall illustrate them.

The nilpotent Lie group $\mathbb{N}il^4$ can be presented as the split extension $\mathbb{R}^3 \rtimes_U \mathbb{R}$ of $\mathbb{R}^3$ by $\mathbb{R}$, where the real three-dimensional representation U of $\mathbb{R}$ has the form $U(t) = \exp(tB)$ with

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the same way, the soluble Lie groups $\mathbb{S}ol^4_{m,n} = \mathbb{R}^3 \rtimes_{T_{m,n}} \mathbb{R}$ forms on real three-dimensional representations $T_{m,n}$ of $\mathbb{R}$, $T_{m,n}(t) = \exp(tC_{m,n})$, where $C_{m,n} = \mathrm{diag}(\alpha, \beta, \gamma)$ and $\alpha+\beta+\gamma = 0$ for $\alpha > \beta > \gamma$. Furthermore, e^α, e^β and e^γ are the roots of $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$, with

m, n the positive integers. If $m = n$, then $\beta = 0$ and $\mathbb{S}ol^4_{m,n} = \mathbb{S}ol^3 \times \mathbb{R}$. In general, if $C_{m,n} \propto C_{m',n'}$, then $\mathbb{S}ol^4_{m,n} \cong \mathbb{S}ol^4_{m',n'}$. It gives infinitely many classes of equivalence. When $m^2n^2 + 18 = 4(m^3 + n^3) + 27$, one has a new geometry, $\mathbb{S}ol^4_0$, associated with the group $\mathbf{SO}(2)$ of isometries rotating the first two coordinates. The soluble group $\mathbb{S}ol^4_1$ is most conveniently represented as the matrix group

$$\left\{ \begin{pmatrix} 1 & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix} : \alpha, a, b, c \in \mathbb{R}, \alpha > 0 \right\}.$$

Finally, the geometry F^4 is associated with the isometry group $\mathbb{R}^2 \rtimes \mathbf{PSL}(2, \mathbb{R})$ and stabilizer $\mathbf{SO}(2)$. Here the semidirect product is taken with respect to the action of the group $\mathbf{PSL}(2, \mathbb{R})$ on $\mathbb{R}^2$. The space F^4 is diffeomorphic to $\mathbb{R}^4$ and has alternating signs of metric. A connection of these geometries with complex and Kählerian structures (preserved by the stabilizer Γ_σ) can be found in [29].

3. KK-groups and K-amenable groups

In this section, we discuss K-groups of the twisted group C^*-algebras, which are relevant to the definition of a twisted analogue of the Kasparov map, pairing between K-theory and cyclic cohomology theory, and which enables us to use K-amenability results for Eilenberg–MacLane spaces.

3.1. KK-groups

Given a manifold X, let $C(X)$ be the commutative C^*-algebra (recall that a C^*-algebra is a Banach algebra with an involution satisfying the relation $\|aa^*\| = \|a\|^2$) of all continuous complex-valued functions which vanish at infinity on X. The C^*-algebra, which categorically encodes the topological properties of manifold X, plays a dual role to X in the K-theory of X by the Serre–Swan theorem [30]:

$$\widetilde{K}^\ell(X) \cong \widetilde{K}_\ell(C(X)), \qquad \ell = 0, 1. \tag{3.1}$$

Here, $\widetilde{K}^\ell(X)$ is the reduced topological K-theory of X: taking into account that a vector bundle over a point is just a vector space, $K(\mathrm{pt}) = \mathbb{Z}$, we can introduce a reduced K-theory in which the topological space consisting of a single point has trivial cohomology, $\widetilde{K}(\mathrm{pt}) = 0$, and also $\widetilde{K}(X) = 0$ for any contractible space X. Let us consider the collapsing and inclusion maps: $p : X \to \mathrm{pt}, \iota : \mathrm{pt} \hookrightarrow X$ for a fixed base point of X. These maps induce an epimorphism and a monomorphism of the corresponding K-groups: $p^* : K(\mathrm{pt}) = \mathbb{Z} \to K(X), \iota^* : K(X) \to K(\mathrm{pt}) = \mathbb{Z}$. The exact sequences of groups are

$$0 \to \mathbb{Z} \xrightarrow{p^*} K(X) \to \widetilde{K}(X) \to 0, \qquad 0 \to \widetilde{K}(X) \to K(X) \xrightarrow{\iota^*} \mathbb{Z}.$$

The kernel of the map i^* (or the cokernel of the map p^*) is called the *reduced K-theory group* and is denoted by $\widetilde{K}(X)$, $\widetilde{K}(X) = \ker \iota^* = \mathrm{coker}\, p^*$. There is a fundamental decomposition $K(X) = \mathbb{Z} \oplus \widetilde{K}(X)$. When X is not compact, we can define $K^c(X)$, the K-theory with compact support. It is isomorphic to $\widetilde{K}(X)$. Let X be a $Spin^{\mathbb{C}}$-manifold, then there is a Poincaré duality isomorphism [31]:

$$K^{\dim X - \ell}(X) \cong K^c_\ell(X), \qquad \ell = 0, 1, \tag{3.2}$$

where K^c_ℓ denotes the dual compactly supported K-homology of X.

For a finite-dimensional manifold X there exists another C^*-algebra, which is noncommutative and can be constructed with the help of the Riemannian metric g. In fact, we can form the complex Clifford algebra $\mathrm{Cliff}(T_xX, g_x)$, where for each $x \in X$ the tangent space T_xX of X is a finite-dimensional Euclidean space with inner product g_x. This algebra has a canonical structure as a finite-dimensional $\mathbb{Z}_2$-graded C^*-algebra. The family of C^*-algebras $\{\mathrm{Cliff}(T_xX, g_x\}_{x\in X}$ forms a $\mathbb{Z}_2$-graded C^*-algebra vector bundle $\mathrm{Cliff}(T_xX) \to X$, called *the Clifford algebra bundle* of X [32]. Let us define $\mathfrak{C}(X) = C(X, \mathrm{Cliff}(T_xX))$ to be the C^*-algebra of continuous sections of the Clifford algebra bundle of X vanishing at infinity. If the manifold X is even-dimensional and has a $Spin^{\mathbb{C}}$-structure then this C^*-algebra is Morita equivalent to $C(X)$. However, in general, $\mathfrak{C}(X)$ is Morita equivalent to $C(T_xX)$. Because of the Morita equivalence of K-theory, it follows that $K_\ell(\mathfrak{C}(X)) \cong K_\ell(C(X)) \cong K^\ell(X)$, $\ell = 0, 1$. But for odd-dimensional and spin manifold X, this relation is more complicated.

Recall that the definition of K-homology involves classifying extensions of the algebra of continuous functions $C(X)$ on the manifold X by the algebra of compact operators up to unitary equivalence [33]. The set of homotopy classes of operators defines the K-homology group $K_0(X)$, and the duality with K-theory is provided by the natural bilinear pairing $([E], [\mathfrak{D}]) \mapsto \mathrm{Index}\mathfrak{D}_E \in \mathbb{Z}$, where $[E] \in K(X)$ and $\mathfrak{D}_E$ denotes the action of the Fredholm operator $\mathfrak{D}$ on the Hilbert space $\mathfrak{H} = L^2(U(X, E))$ of square-integrable sections of the vector bundle $E \to X$ as $\mathfrak{D} : U(X, E) \to U(X, E)$. It assumes that the KK-pairing may be the most natural framework in this context. The group $KK(A, B)$ is a bivariant version of K-theory and depends on a pair of graded algebras A and B.[3]

Definition 3.1. *Let A and B be C^*-algebras.*

- $\mathscr{D}$*1. A pair $(\mathcal{E}, \pi)$ will be called an (A, B)-bimodule if $\mathcal{E}$ is a $\mathbb{Z}/2\mathbb{Z}$ graded Hilbert B-module on which algebra A acting by means of $*$-homomorphism π: $A \to \mathfrak{L}(\mathcal{E}) = \mathrm{End}^*(\mathcal{E})$, where $\forall a \in A$ an operator $\pi(a)$ being of degree 0, $\pi(A) \subset \mathfrak{L}(\mathcal{E})^{(0)}$. Let $E(A, B)$ be a triple $(\mathcal{E}, \pi, F)$, where $(\mathcal{E}, \pi)$ is a A, B-bimodule, $F \in \mathfrak{L}(\mathcal{E})$ is a homogeneous operator of degree 1, and $\forall a \in A$: $\pi(a)(F^2 - 1) \in C(\mathcal{E})$, $[\pi(a), F] \in C(\mathcal{E})$, where $C(\mathcal{E})$ is the algebra of compact operators.*
- $\mathscr{D}$*2. A triple $(\mathcal{E}, \pi, F)$ will be called degenerate if $\forall a \in A$: $\pi(a)(F^2-1) = 0$, $[\pi(a), F] = 0$.*
- $\mathscr{D}$*3. Let $D(A, B)$ be a set of degenerated triples. An element $E(A, B[0, 1])$, where $B[0, 1]$ is an algebra of continuous functions in B on the interval* $[0, 1]$*, will be called a homotopy in $E(A, B)$.*

Let us assign a direct sum in $E(A, B)$:

$$(\mathcal{E}, \pi, F) \oplus (\mathcal{E}', \pi', F') = (\mathcal{E} \oplus \mathcal{E}', \pi \oplus \pi', F \oplus F'). \tag{3.3}$$

The homotopy classes of $E(A, B)$ together with this sum defines the Abelian group $KK(A, B)$. It is clear that any degenerate triplet is homotopy equivalent to $(0, 0, 0)$, and the inverse element to $(\mathcal{E}, \pi, F)$ is equal to $(-\mathcal{E}, \pi, -F)$, where $-\mathcal{E}$ means that grading on $\mathcal{E}$ has to be inverted. $f : A_1 \to A_2$ transfers (A_2, B)-modules into (A_1, B)-modules and [36]

$$f^* : E(A_2, B) \to E(A_1, B), \qquad (\mathcal{E}, \pi, F) \mapsto (\mathcal{E}, \pi^\circ f, F). \tag{3.4}$$

On the other hand, a $*$-homomorphism $g : B_1 \to B_2$ induces a homomorphism

$$g_* : E(A, B_1) \to E(A, B_2), \qquad (\mathcal{E}, \pi, f) \mapsto (\mathcal{E} \otimes_g B_2, \pi \otimes 1, F \otimes 1), \tag{3.5}$$

where

$$\pi \otimes 1 : A \to \mathcal{L}(\mathcal{E} \otimes_g B_2), \qquad (\pi \otimes 1)(a)(e \otimes b) = \pi(a)e \otimes b. \tag{3.6}$$

[3] The reader can find the application of KK-group theory to the classification of branes, for example, in [34, 35].

Theorem 3.1. *The groups $KK(A,B)$ define an homotopy invariant bifunctor from the category of separable C^*-algebras into the category of Abelian groups. Abelian groups $KK(A,B)$ depend contravariantly on the algebra A and covariantly on the algebra B, in addition $KK(\mathbb{C},B)=K_0(B)$.*[4]

Definition 3.2. *Let $1_A\in KK(A,A)$ ($KK(A,A)$ is a ring with unit) denotes the triple class $(A,\iota_A,0)$, where $A^{(1)}=A$, $A^{(0)}=0$ and $\iota_A:A\to C(A)\subset\mathfrak{L}(A)$, $\iota_A(a)b=ab, a,b\in A$. Let us define also the following map:*

$\mathscr{D}1.$ $\tau_D : KK(A,B)\otimes KK(A\otimes D, B\otimes D)$.
$\mathscr{D}2.$ $\tau_D(\text{class}(\mathcal{E},\pi,F))=\text{class}(\mathcal{E}\otimes D,\pi\otimes 1_D, F\otimes 1)$.

Theorem 3.2. *Kasparov's pairing, defined by*

$$KK(A,D)\times KK(D,B)\longrightarrow KK(A,B) \tag{3.7}$$

and denoted $(x,y)\mapsto x\otimes_D y$, satisfies the following properties:

$\mathscr{I}1.$ *It depends covariantly on the algebra B and contravariantly on the algebra A.*
$\mathscr{I}2.$ *If $f:D\to E$ is a $*$-homomorphism, then $f_*(x)\otimes_E y = x\otimes_D f^*(y), x\in KK(A,D), y\in KK(E,B)$.*
$\mathscr{I}3.$ *Associative property: $(z\otimes_D y)\otimes_E z = x\otimes_D(y\otimes)_E z, \forall x\in KK(A,D), y\in KK(D,E), z\in KK(E,B)$.*
$\mathscr{I}4.$ $x\otimes_B 1_B = 1_A\otimes x = x, \forall x\in KK(A,B)$.
$\mathscr{I}5.$ $\tau_E(x\otimes_B y)=\tau_E(x)\otimes_{B\otimes E}\tau_E(y), \forall x\in KK(A,B), \forall y\in KK(B,D)$.

Suppose that for two algebras A and B, there are elements $\alpha\in KK(A\otimes B,\mathbb{C}), \beta\in KK(\mathbb{C},A\otimes B)$, with the property that $\beta\otimes_A\alpha = 1_B\in KK(B,B), \beta\otimes_B\alpha=1_A\in KK(A,A)$. Then we say that we have KK-duality isomorphisms between the K-theory (K-homology) of the algebra A and the K-homology (K-theory) of the algebra B

$$K_*(A)\cong K^*(B),\qquad K^*(A)\cong K_*(B). \tag{3.8}$$

In fact, the algebras A and B are Poincaré dual [37], but generally speaking these algebras are not KK-equivalent.

3.2. K-amenable groups

We now review the concept of K-amenable groups [15]. Let G be a connected Lie group and K a maximal compact subgroup. We also assume that $\dim(G/K)$ is even and G/K admits a G-invariant $Spin^{\mathbb{C}}$-structure. The G-invariant Dirac operator $\mathfrak{D}:=\gamma^\mu\partial_\mu$ on G/K is a first-order self-adjoint, elliptic differential operator acting on L^2- sections of the $\mathbb{Z}_2$-graded homogeneous bundle of spinors $\mathfrak{S}$. Let us consider a zeroth-order pseudo-differential operator $\mathscr{O}=\mathfrak{D}(1+\mathfrak{D}^2)^{-\frac{1}{2}}$ acting on $H=L^2(G/K,\mathfrak{S})$. $C(G/K)$ acts on H by multiplication of operators. G acts on $C(G/K)$ and on H by left translation, and $\mathscr{O}$ is G-invariant. Then, the set $(\mathscr{O},H,X)$ defines a canonical Dirac element $\alpha_G=KK_G(C(G/K),\mathbb{C})$.

Theorem 3.3 (*Kasparov* [38])**.** *There is a canonical Mishchenko element*

$$\alpha_G\in KK_G(C(G/K),\mathbb{C}) \tag{3.9}$$

such that the following intersection products occur:

$\mathscr{I}1.$ $\alpha_G\otimes_{\mathbb{C}}\beta_G = 1_{C(G/K)}\in KK_G(C(G/K),C(G/K))$.

[4] The interesting facts to us are the following relations: $KK^*(A:=\mathbb{C},B)=K_*(B)$, $KK^*(A,B:=\mathbb{C})=K^*(A)$.

$\mathscr{I}$2. $\beta_G \otimes_{C(G/K)} \alpha_G = \gamma_G = KK_G(\mathbb{C}, \mathbb{C})$ where γ_G is an element in $KK_G(\mathbb{C}, \mathbb{C})$.

All solvable groups are amenable, while any non-compact semisimple Lie group is non-amenable. For a semisimple Lie group G or for $G = \mathbb{R}^n$, a construction of the Mishchenko element β_G can be found in [15]. We now come to the basic theorem and definition.

Theorem 3.4 (*Kasparov* [38])**.** *If group G is amenable, then $\gamma_G = 1$.*

Definition 3.3. *A Lie group G is said to be K-amenable if γ_G=1.*

Proposition 3.1. *The following statements hold:*

$\mathscr{P}$1. Any solvable Lie group and any amenable Lie group is K-amenable.
$\mathscr{P}$2. The non-amenable groups $\mathbf{SO}_0(n, 1)$ are K-amenable Lie groups [16].
$\mathscr{P}$3. The groups $\mathbf{SU}(n, 1)$ are K-amenable Lie groups [39].
$\mathscr{P}$4. The class of K-amenable groups is closed under the operations of taking subgroups, under free and direct products [40].

4. Aspects of K-group theory

4.1. K-theory of twisted group C^*-algebras

4.1.1. Twisted crossed products of C^*-algebras[5]. Let us consider a general family of twisted actions of locally compact groups on C^*-algebras, and the corresponding twisted crossed product C^*-algebras. We start with the definition of a twisted action of a locally compact group G on C^*-algebra A (see for detail [12]). Let Aut$A$ and $UM(A)$ denote its automorphism group and the group of unitary elements in its multiplier algebra $M(A)$. A twisted action of G on A is a pair α, u of Borel maps $\alpha : G \to \mathrm{Aut}A, u : G \times G \to UM(A)$ satisfying $\alpha_s \circ \alpha_t = \mathrm{Ad}u(s,t) \circ \alpha_{st}, \alpha_r(u(s,t))u(r,st) = u(r,s)u(rs,t)$. These twisted actions and a twisted Banach $*$-algebra $L^1(A, G, \alpha, u)$ have been introduced in [41]. The quadruple (A, G, α, u) can be referred as a *(separable) twisted dynamical system*:

- The covariant theory of the system (A, G, α, u) can be realized on Hilbert space. The corresponding reduced crossed product can be defined as the C^*-algebra generated by the regular representation [12]. In [42] a duality theorem has been proved for that C^*-algebra.
- The twisted cross product $A \times_{\alpha,u} G$ was defined as a C^*-algebra whose representation theory is the same as the covariant representation theory of (A, G, α, u) on Hilbert space [12]. A cross product by coactions of possibly non-amenable groups also has been considered in [43].
- Suppose that (α, u) is a twisted action of an amenable group G on a C^*-algebra A which is the algebra of sections of a C^*-bundle $\mathscr{E}$ over X; each α leaves all ideals $I_x = \{a \in A : a(x) = 0\}$ invariant. It has been shown [13] that $A \times_{\alpha,u} G$ is the algebra of sections of a C^*-bundle over X with fibres of the form $(A/I_x) \times_{\alpha(x),u(x)} G$. This result has been proved by using the stabilization trick of [12].

  Stabilization trick. The twisted cross product algebra $A \rtimes_\sigma \Gamma$ is stably equivalent to the cross product $(A \otimes \mathscr{K}) \rtimes \Gamma$, where $\mathscr{K}$ denotes the compact operators (we refer the reader to the article [12] for details).

[5] It is often useful to introduce C^*-algebras A as involute Banach algebras for which the following equalities hold $(xy)^* = y^*x^*$, $\|x^*x\| = \|x\|^2$ for $x, y \in A$. A unique norm is given for any x by $\|x\|$ = (spectral radius of $(x^*x)^{1/2}$).

4.1.2. Twisted K-groups. Let Γ be a discrete cocompact subgroup of a solvable simply connected Lie group G. It has been shown in [14] that

$$K_*(C^*(\Gamma)) \cong K^{*+\dim G}(G/\Gamma). \tag{4.1}$$

For a multiplier σ on Γ (a cocycle $\sigma \in Z^2(\Gamma, \mathbf{U}(1))$), the K-theory of the twisted group algebra $C^*(\Gamma, \sigma)$ is that of a continuous-trace C^*-algebra B_σ with spectrum G/Γ, i.e. the twisted K-theory $K^*(G/\Gamma, \delta(B_\sigma))$ [13]. The Dixmier–Douady class $\delta(B_\sigma)$ can be identified as the image of σ under a homomorphism [44, 45]:

$$\delta : H^2(\Gamma, \mathbf{U}(1)) \longrightarrow H^3(G/\Gamma, \mathbb{Z}), \tag{4.2}$$

it depends only on the homotopy class of σ in $H^2(\Gamma, \mathbf{U}(1))$. This result has been extended in [13] to describe the K-theory of the twisted transformation group algebras $C_0(X) \times_{\tau,\omega} \Gamma$, where X is a Γ-space and $\omega \in Z^2(\Gamma, C(X, \mathbf{U}(1)))$. Thus, $K_*(C_0(X) \times_{\tau,\omega} \Gamma)$ is isomorphic to a twisted K-group $K^*((G \times X)/\Gamma, \delta(\omega))$ of the orbit space $(G \times X)/\Gamma$ for the diagonal action, and identify the twist $\delta(\omega) \in H^3((G \times X)/\Gamma, \mathbb{Z})$. Further generalization can be obtained for describing the twisted transformation group algebra $C_0(Y) \times_{\tau,\omega} G$ associated with a locally compact group G, a principal G-bundle Y and cocycle $\omega \in Z^2(G, C(Y, \mathbf{U}(1)))$ [13].

Let $\Gamma \subset G$ be a lattice in G and A be an algebra admitting an automorphic action of Γ. The cross product algebra $[A \otimes C_0(G/K)] \rtimes \Gamma$ is Morita equivalent to the algebra of continuous sections vanishing at infinity $C_0(\Gamma\backslash G/K, E)$. Here, $E \to \Gamma\backslash G/K$ is the flat A-bundle defined as the quotient

$$E = (A \times G/K)/\Gamma \longrightarrow \Gamma\backslash G/K, \tag{4.3}$$

and we consider the diagonal action of Γ on $A \times G/K$.

Theorem 4.1 (*Kasparov* [38]). *Let G be a K-amenable, then $(A \rtimes \Gamma) \otimes C_0(G/K)$ and $[A \otimes C_0(G/K)] \rtimes \Gamma$ have the same K-theory.*

Corollary 4.1. *Let G be a K-amenable, then $(A \rtimes \Gamma) \otimes C_0(G/K)$ and $C_0(\Gamma\backslash G/K, E)$ have the same K-theory. It means that for $\ell = 0, 1$ one has*

$$K_\ell(C_0(\Gamma\backslash G/K, E)) \cong K_{\ell+\dim(G/K)}(A \rtimes \Gamma). \tag{4.4}$$

We follow the lines of the article [15] in the formulation and in the proof of the main theorem which generalizes theorems of [12] and [13].

Theorem 4.2 (Carey *et al* [15]). *Let Γ be a lattice in a K-amenable Lie group G and K be a maximal compact subgroup of G. Then*

$$K_*(C^*(\Gamma, \sigma)) \cong K^{*+\dim(G/K)}(\Gamma\backslash G/K, \delta(B_\sigma)), \tag{4.5}$$

where $\sigma \in H^2(\Gamma, U(1))$ is any multiplier on Γ, $K^*(\Gamma\backslash G/K, \delta(B_\sigma))$ is the twisted K-theory of a continuous trace C^*-algebra B_σ with spectrum $\Gamma\backslash G/K$ and $\delta(B_\sigma)$ denotes the Dixmier–Douady invariant of B_σ.

Proof. First suppose that $A = \mathbb{C}$ and Γ acting trivially on $\mathbb{C}$. Because of corollary 4.1 when $\gamma_G = 1$ one gets

$$\underbrace{\begin{Bmatrix} (A \rtimes \Gamma) \otimes C_0(G/K) \\ C_0(\Gamma\backslash G/K, E) \end{Bmatrix}}_{\text{have the same K-theory}} = \underbrace{\begin{Bmatrix} (\mathbb{C} \rtimes \Gamma) \otimes C_0(G/K) \\ C_0(\Gamma\backslash G/K, E) \end{Bmatrix}}_{\text{have the same K-theory}} \overset{\text{theorem 4.1}}{\Longleftrightarrow} \underbrace{\begin{Bmatrix} [\mathbb{C} \otimes C_0(G/K)] \rtimes \Gamma \\ C_0(G/K, E) \end{Bmatrix}}_{\text{have the same K-theory}}$$

$$\Longrightarrow C^*(\Gamma) \text{ and } C_0(\Gamma\backslash G/K) \text{ have the same K-theory.}$$

Table 3. $\dim(G/K) = 3$; Dixmier–Douady invariant and cohomologies.

$K^*(\delta) \equiv K^*(\Gamma\backslash G/K, \delta(B_\sigma))$ (G is K-amenable)	$\delta(B_\sigma)$	$H^3(\Gamma\backslash G/K, \mathbb{Z})$
Non-twisted group $K^*(0)$ ($\Gamma\backslash G/K$ is orientable)	0 (B_σ is Morita equivalent to $C(\Gamma\backslash G/K)$	$\mathbb{Z}$
Non-twisted group $K^*(0)$ ($\Gamma\backslash G/K$ is not orientable)	0 (B_σ is Morita equivalent to $C(\Gamma\backslash G/K)$	0
Twisted group $K^*(\delta)$	$\neq 0$ ($K^*(\delta)$ can be not isomorphic to $K^*(0)$, but there is isomorphism for $\delta = 0$ [12])	

Suppose $\sigma \in H^2(\Gamma, \mathbf{U}(1))$. If G is K-amenable, then using corollary 4.1 one sees

$$\underbrace{\left\{\begin{matrix}(A \rtimes \Gamma) \otimes C_0(G/K) \\ C_0(\Gamma\backslash G/K, E)\end{matrix}\right\}}_{\text{have the same K-theory}} \Longrightarrow \underbrace{\left\{\begin{matrix}(A \rtimes_\sigma \Gamma) \otimes C_0(G/K) \\ C_0(\Gamma\backslash G/K, E_\sigma)\end{matrix}\right\}}_{\text{have the same K-theory}},$$

$$\text{where } E_\sigma = (A \otimes \mathscr{K} \times G/K)/\Gamma \longrightarrow \Gamma\backslash G/K.$$

Since by definition the twisted K-theory $K^*(\Gamma\backslash G/K, \delta(B_\sigma))$ is the K-theory of the continuous trace C^*-algebra $B_\sigma = C_0(\Gamma\backslash G/K, E_\sigma)$ with spectrum $\Gamma\backslash G/K$. Thus equation (4.5) follows. □

4.2. The three-dimensional case

One of the main results of [15] says that for lattices in K-amenable Lie groups the reduced and unreduced twisted group C^*-algebras have canonically isomorphic K-theory. If $\sigma \in H^2(\Gamma, \mathbf{U}(1))$ is a multiplier on Γ and Γ is a lattice in a K-amenable Lie group, then the canonical morphism $C^*(\Gamma, \sigma) \to C_r^*(\Gamma, \sigma)$ induces an isomorphism

$$K_*(C^*(\Gamma, \sigma)) \cong K_*(C_r^*(\Gamma, \sigma)). \tag{4.6}$$

Corollary 4.2. *Suppose G is a connected Lie group and K a maximal compact subgroup such that* $\dim(G/K) = 3$. *Let* Γ *be a uniform lattice in G and* $\sigma \in H^2(\Gamma, U(1))$ *be any multiplier on* Γ. *Suppose also that G is K-amenable, then for* $\ell = 0, 1$ *(mod 2)*

$$K_\ell(C_r^*(\Gamma, \sigma)) \cong K_\ell(C_r^*(\Gamma)) \cong K^{\ell+1}(\Gamma\backslash G/K). \tag{4.7}$$

Indeed, by the main theorem 4.2, $K_\ell(C_r^*(\Gamma)) \cong K^{\ell+\dim(G/K)}(\Gamma\backslash G/K)$, for $\ell = 0, 1$ (mod 2). Because of the Packer–Raeburn stabilization trick, $C_r^*(\Gamma, \sigma)$ is Morita equivalent to $C \rtimes \Gamma$, and since G is K-amenable, $C \rtimes \Gamma \otimes C_0(G/K)$ is Morita equivalent to $B_\sigma = C(\Gamma\backslash G/K, E_\sigma)$. Here as before E_σ is a locally trivial bundle of C^*-algebras over $\Gamma\backslash G/K$ with fibre $\mathscr{K}$. For the Dixmier–Douady invariant, one has $\delta(B_\sigma) = \delta(\sigma) \in H^3(\Gamma\backslash G/K, \mathbb{Z}) \cong H^3(\Gamma, \mathbb{Z})$. If $\Gamma\backslash G/K$ is not orientable, then $H^3(\Gamma\backslash G/K, \mathbb{Z}) = \{0\}$. Therefore $\delta(B_\sigma) = 0$ (see table 3) and equation (4.7) holds. On the contrary, when $\Gamma\backslash G/K$ is orientable, $\delta(\sigma) = 0$ for all $\sigma \in H^2(\Gamma, U(1))$ [15], and B_σ is Morita equivalent to $C(\Gamma\backslash G/K)$ (see table 3). In this case again we have equation (4.7).

Corollary 4.3 (Carey *et al* [15])**.** *Let* $M = K(\Gamma, 1)$ *be an Eilenberg–MacLane space which is a connected locally symmetric, compact, three-dimensional manifold. Let* $\sigma \in H^2(\Gamma, U(1))$ *be any multiplier on* Γ*, then one has*

$$K_\ell(C_r^*(\Gamma, \sigma)) \cong K_\ell(C_r^*(\Gamma)) \cong K^{\ell+1}(M), \qquad \ell = 0, 1. \tag{4.8}$$

Indeed, $K(\Gamma, 1)$ is locally symmetric (see the appendix for the necessary information on Eilenberg–MacLane spaces), and therefore is of the form $\Gamma\backslash G/K$, where G is a connected Lie group, K is a maximal compact subgroup such that $\dim(G/K) = 3$ and $\Gamma \subset G$ is a uniform lattice in G. According to Thurston's list of three-geometries or locally homogeneous spaces, one has

- $G = \mathbb{R}^3 \rtimes \mathbf{SO}(3)$, $G/K = \mathbb{R}^3$ (flat).
- $G = \mathbf{SO}_0(3, 1)$, $G/K = \mathbb{H}^3$ (hyperbolic; compact example: nontrivial $\Sigma_{g\geqslant 2}$-bundle over S^1.
- $G = \mathbf{SO}_0(2, 1) \rtimes \mathbb{R}$, $G/K = \mathbb{H}^2 \times \mathbb{R}$ (hyperbolic; compact example: trivial $\Sigma_{g\geqslant 2}$-bundle over S^1, i.e. $\Sigma_g \times S^1$).
- $G = \mathbb{N}il^3 = \mathbb{Z} \rtimes \mathbb{R}^2$ (central, non-split extension), $G/K = \mathbb{N}il^3$ (flat; compact example: $\mathbb{T}^2$-bundle over S^1 via mapping torus).
- $G = \mathbb{S}ol^3 = \mathbb{R}^2 \rtimes \mathbb{R}$ (split extension), $G/K = \mathbb{S}ol^3$ (flat; compact example: $\mathbb{T}^2$-bundle over S^1 via mapping torus).
- $G = \widetilde{\mathbf{SO}_0(2, 1)} \rtimes \mathbb{R}$, $G/K = \widetilde{\mathbf{SO}_0(2, 1)}$ (hyperbolic; compact example: S^1-bundle over $F_{g\geqslant 2}$).

For all of these three-manifolds $\gamma_G = 1$. More two locally homogeneous spaces in Thurston's list are not locally symmetric. We can apply corollary 4.2 in order to deduce corollary 4.3.

4.3. The four-dimensional case

Using the mathematical tools exposed so far, we are now able to formulate new results concerning K-amenability in the four-dimensional case.

Corollary 4.4. *Lie groups G associated with spaces $G/K = \mathbb{R}^4, \mathbb{H}^4, \mathbb{C}P^2, \mathbb{C}\mathbb{H}^2, \mathbb{H}^2 \times \mathbb{R}^2, \mathbb{H}^2 \times \mathbb{H}^2, \mathbb{H}^3 \times \mathbb{R}, \mathbb{N}il^3 \times \mathbb{R}, \mathbb{S}ol^4, \mathbb{N}il^4, \mathbb{S}ol^4_{m,n}$ (including $\mathbb{S}ol^3 \times \mathbb{R}$), $\mathbb{S}ol^4_1, \widetilde{\mathbf{PSL}(2, \mathbb{R})} \times \mathbb{R}, F^4$ enumerated in table 2 are K-amenable.*

Proof. We need to prove that $\gamma_G = 1$. According to the list of four-geometries, table 2, one has the following result:

$\mathscr{C}1$. $G = \mathbb{R}^4 \rtimes \mathbf{SO}(4)$, $G/K = \mathbb{R}^4$, $\gamma_G = 1$ since $\mathbb{R}^4$ and $\mathbf{SO}(4)$ are amenable, and so is their semidirect product (proposition 3.1).

$\mathscr{C}2$. $G = \mathbf{SO}_0(4, 1)$, $G/K = \mathbb{H}^4$, $\gamma_G = 1$ by Kasparov's theorem (proposition 3.1).

$\mathscr{C}3$. $G = \mathbf{SU}(3)$, $G/K = \mathbb{C}P^2 \simeq \mathbf{U}(3)/(\mathbf{U}(1) \times \mathbf{U}(2)) \simeq \mathbf{SU}(3)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(2))$, $\gamma_G = 1$ (proposition 3.1).

$\mathscr{C}4$. $G = G/K = \mathbb{C}\mathbb{H}^2$, the geometry $\mathbb{C}\mathbb{H}^2$ is a Kählerian symmetric space and certainly carries a complex structure, $\gamma_G = 1$.

$\mathscr{C}5$. $G/K = \mathbb{H}^2 \times \mathbb{R}^2, \mathbb{H}^2 \times \mathbb{H}^2, \mathbb{H}^3 \times \mathbb{R}$, $\gamma_G = 1$ since these groups are free products of K-amenable groups (proposition 3.1).

$\mathscr{C}6$. $G/K = \mathbb{N}il^3 \times \mathbb{R}, \mathbb{S}ol^4, \mathbb{N}il^4, \mathbb{S}ol^4_{m,n}$ (including $\mathbb{S}ol^3 \times \mathbb{R}$), $\mathbb{S}ol^4_1$ $\gamma_G = 1$ since nilpotent and solvable groups are K-amenable groups and so is their semidirect product (proposition 3.1).

$\mathscr{C}7$. $G/K = \widetilde{\mathbf{PSL}(2, \mathbb{R})} \times \mathbb{R}$, group $\widetilde{\mathbf{PSL}(2, \mathbb{R})}$ is diffeomorphic to $\mathbb{R}^3$, $\gamma_G = 1$ since X is free products of K-amenable groups (proposition 3.1).

$\mathscr{C}8$. $G/K = F^4$. One can choose a subgroup $\mathscr{G}_X$ of X which admits X as a principal homogeneous space [29]. Take $\mathscr{G}_X = \mathbb{R}^2 \ltimes \mathbf{SL}(2, \mathbb{R})$ (with the natural action of $\mathbf{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$) consist of the upper triangular matrices with positive diagonal entries. $\gamma_G = 1$ because of the semidirect product of K-amenable groups (proposition 3.1). □

The other five locally homogeneous spaces $\mathbb{S}^4, \mathbb{S}^2 \times \mathbb{R}^2, \mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2 \times \mathbb{H}^2, \mathbb{S}^3 \times \mathbb{R}$ in Thurston's list are not locally symmetric spaces.

Remark 4.1. Are corollaries 4.3 and 4.4 still valid without the locally symmetric assumption on $K(\Gamma, 1)$? The answer of this interesting question was found in [15] where the interesting conjecture has been formulated without the locally symmetric assumption on three-manifolds: suppose $K(\Gamma, 1)$ is a connected, compact, three-manifold which is an Eilenberg–MacLane space with fundamental group Γ. Then for any multiplier $\sigma \in H^2(\Gamma, U(1))$ on Γ, $K_j(C_r^*(\Gamma, \sigma)) \cong K_j(C_r^*(\Gamma)) \cong K^{j+1}(K(\Gamma, 1))$, $j = 0, 1$.

We can generalize this statement for the case of four-manifolds in terms of the following conjecture.

Conjecture 4.1. *Let $K(\Gamma, 1)$ be a connected, compact, four-dimensional manifold which is an Eilenberg–MacLane space with fundamental group Γ. For any multiplier $\sigma \in H^2(\Gamma, U(1))$ on Γ, one has*[6]

$$K_j(C_r^*(\Gamma, \sigma)) \cong K_j(C_r^*(\Gamma)) \cong K^{j+1}(K(\Gamma, 1)), \qquad j = 0, 1. \tag{4.9}$$

5. T-duality and topology change from fluxes

First we discuss T-duality in one direction only; then T-dualizing on the torus will be considered in the following sections. A more general case with T-dualizing on $\mathbb{T}^n$, $n > 2$ can be obtained by appropriate successive dualizations [22]. The application of T-duality is not restricted to product manifolds $M \times S^1$, but can also be applied locally in the case of S^1-fibrations over M [46], and can be generalized to situations with nontrivial NS three-form flux H. A more general case where X is an oriented S^1-bundle over the manifold M characterizes by its first Chern class $c_1(X) \in H^2(M, \mathbb{Z})$ in the presence of (possibly nontrivial) H-flux $\delta(B) \equiv [H] \in H^3(X, \mathbb{Z})$.[7]

Remark 5.1. To simplify notations we will use the same notation for a cohomology class $[H]$, or for a representative H, throughout the last part of this paper. For the reader it should be clear which is meant from the context.

It has been argued [22] that the T-dual of X, $\widehat{X}$, is again an oriented $\widehat{S}^1$-bundle over $\widehat{M}$

$$\begin{array}{ccccc} S^1 \longrightarrow & X & & \widehat{S}^1 \longrightarrow & \widehat{X} \\ & \pi\big\downarrow & \xleftarrow{\text{T-duality}}\!\!\longrightarrow & & \hat{\pi}\big\downarrow \\ & M & & & \widehat{M} \end{array} \tag{5.1}$$

supporting H-flux $\widehat{H} \in H^3(\widehat{X}, \mathbb{Z})$, and $c_1(X) = \pi_* H$, $c_1(\widehat{X}) = \widehat{\pi}_* \widehat{H}$. Here, $\pi_* : H^k(X, \mathbb{Z}) \to H^{k-1}(M, \mathbb{Z})$, and similarly $\widehat{\pi}_*$, denotes the pushforward maps. (As an example, at the level of the de Rham cohomology the pushforward maps π_* and $\widehat{\pi}_*$ are simply the integrations along the S^1-fibres ($\widehat{S}^1$-fibres) of X ($\widehat{X}$).)

6 Probably the Dixmier–Douady invariant $\delta(\sigma) = 0$ for all $\sigma \in H^2(\Gamma, U(1))$ for Γ as in the conjecture.

7 Let us consider a bundle whose fibre F is $(p-1)$-connected; this means that for $k < p$ the kth homotopy group $\pi_{k<p}(F)$ of F vanishes. Let the pth homotopy group be nontrivial and is equal to the group $G, \pi_{k=p}(F) = G$. Then F bundles can be characterized by a degree $(p+1)$ characteristic class in the cohomology with coefficients in $G, \omega_{p+1} \in H^{p+1}(X : G)$ and some characteristic classes of higher degree. Let all of the homotopy classes of F of degree higher than p vanish, then the bundle are characterized by ω_{p+1}. In particular, for $\mathbb{Z}_2$ bundles $p = 0, G = \mathbb{Z}_2$ and $\omega_1 \in H^1(X; \mathbb{Z}_2)$ is the *spin* structure. Circle bundles, $p = 1, G = \mathbb{Z}$, characterized by a single characteristic class $c_1 = \omega_2 \in H^2(X; \mathbb{Z})$ which is called *the first Chern class.*

Proposition 5.1 (Gysin sequence)**.** *Let $\pi : X \to M$ be an oriented sphere bundle with fibre $\mathbb{S}^k$. Then there is a long exact sequence (see for example [47])*

$$\cdots \longrightarrow H^n(X) \xrightarrow{\pi_*} H^{n-k}(M) \xrightarrow{\wedge e} H^{n+1}(M) \xrightarrow{\pi^*} H^{n+1}(M) \longrightarrow \cdots \tag{5.2}$$

where the maps π_*, $\wedge e$ and π^* are an integration along the fibre, a multiplication by the Euler class and the natural pullback, respectively.

In the case of an oriented S^1 bundle with the first Chern class $c_1(X) = F \in H^2(M, \mathbb{Z})$, one gets

$$\cdots \longrightarrow H^\ell(M, \mathbb{Z}) \xrightarrow{\pi_*} H^\ell(X, \mathbb{Z}) \xrightarrow{\pi_*} H^{\ell-1}(M, \mathbb{Z}) \xrightarrow{F\cup} H^{\ell+1}(M, \mathbb{Z}) \longrightarrow \cdots. \tag{5.3}$$

In particular, for a two-dimensional base manifold M, the Gysin sequence gives an isomorphism between $H^3(X, \mathbb{Z})$ and $H^2(M, \mathbb{Z})$, i.e. between Dixmier–Douady classes on X and line bundles on M. For example, $\mathbb{S}^3$ manifold can be considered as an S^1-bundle over $\mathbb{S}^2$ by means of the Hopf fibration. T-duality in the absence of H-flux leads to $\mathbb{S}^2 \times S^1$ manifold supported by one unit of H-flux [22].

Circle bundles on Riemannian surfaces. Let us consider twisted K-groups of circle bundles over two-manifolds and their T-duals. It can be shown that in this case K^0 of each space is related to K^1 of its dual. This class of manifolds includes:

- The familiar examples of NS5-branes.
- Three-dimensional lens spaces.
- $\mathbb{N}il$-manifolds.

The K-groups are determined by the Atiyah–Hirzebruch spectral sequence. It is convenient to consider the first differential $d_3 = Sq^3 + H$ of the sequence only. If $H^{\text{even}}(X, \mathbb{Z})$ and $H^{\text{odd}}(X, \mathbb{Z})$ are the even and odd cohomology classes of the manifold X, then the twisted K-groups are

$$K^0(X, H) = \frac{\text{Ker}(H\cup : H^{\text{even}} \longrightarrow H^{\text{odd}})}{H \cup H^{\text{odd}}(X, \mathbb{Z})}, \qquad K^1(X, H) = \frac{\text{Ker}(H\cup : H^{\text{odd}} \longrightarrow H^{\text{even}})}{H \cup H^{\text{even}}(X, \mathbb{Z})}. \tag{5.4}$$

The case of noncommutative D2-branes has been considered in [15]. These branes can wrap two-dimensional manifolds, which in the presence of a constant B-field are described by noncommutative Riemann surface. Let $M = \Sigma_g \equiv \mathbb{H}^2/\Gamma_g$ be a Riemann surface of genus g. We can specialize to the case when $G = \mathbb{R}^2$, $K = \{e\}$ and $g = 1$, with Γ_1 being $\mathbb{Z}^2$. Let $\sigma \in H^2(\Gamma_\sigma, \mathbf{U}(1))$ be any multiplier on Γ_σ. The graded groups are given by

$$Gr(K^0(\Sigma_g)) = \oplus_j E_\infty^{2j}(\Sigma_g), \qquad Gr(K^1(\Sigma_g)) = \oplus_j E_\infty^{2j+1}(\Sigma_g). \tag{5.5}$$

In two dimensions, the Chern character is an isomorphism over the integers and therefore we get

$$K^0(\Sigma_g) \cong H^0(\Sigma_g, \mathbb{Z}) \oplus H^2(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^2, \qquad K^1(\Sigma_g) \cong H^1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}. \tag{5.6}$$

Using theorem 4.2, we have

$$K_\ell(C_r^*(\Gamma_g)) \cong K^\ell(\Sigma_g), \qquad K_\ell(C_r^*(\Gamma_g, \sigma)) \cong K_\ell(\Sigma_g, \delta(B_\sigma)), \ell = 0, 1. \tag{5.7}$$

Here, $B_\sigma = C(\Sigma_g, E_\sigma)$. Note that E_σ is a locally trivial flat bundle of C^*-algebras over Σ_g, with fibre $\mathcal{K}$ ($\mathcal{K}$ are compact operators), it has a Dixmier–Douady invariant $\delta(B_\sigma)$ which can be viewed as the obstruction to B_σ being Morita equivalent to $C(\Sigma_g)$ [15]. It is evident that $\delta(B_\sigma) = \delta(\sigma) \in H^3(\Sigma_g, \mathbb{Z}) = 0$. Thus, B_σ is Morita equivalent to $C(\Sigma_g)$ and finally

$$K_j(C_r^*(\Gamma_g, \sigma)) \cong K_j(C_r^*(\Gamma_g)) \cong K^j(\Sigma_g),$$
$$K^0(\Sigma_g) \cong H^0(\Sigma_g, \mathbb{Z}) \oplus H^2(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^2, K^1(\Sigma_g) \cong H^1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}. \tag{5.8}$$

Three-cycles. For a Riemann surface of genus g, $H^2(\Sigma_g, \mathbb{Z}) = \mathbb{Z}$ and topologically circle bundles are classified by an integer j. The cohomology of the total space X^3 are[8]

- $j = 0$ (trivial line bundle): $H^0(X^3, \mathbb{Z}) = \mathbb{Z}, H^1(X^3, \mathbb{Z}) = \mathbb{Z}^{2g+1}, H^2(X^3, \mathbb{Z}) = \mathbb{Z}^{2g+1}, H^3(X^3, \mathbb{Z}) = \mathbb{Z}$;
- $j \neq 0$ (the Chern class equal to j): $H^0(X^3, \mathbb{Z}) = \mathbb{Z}, H^1(X^3, \mathbb{Z}) = \mathbb{Z}^{2g}, H^2(X^3, \mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathbb{Z}_j, H^3(X^3, \mathbb{Z}) = \mathbb{Z}$.

Then the untwisted ($H = 0$) and twisted ($H = k$) K-groups are given by [22]

$$\begin{aligned}
&K^0(X^3, H = 0) = H^0(X^3, \mathbb{Z}) \oplus H^2(X^3, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{2g+2} & \text{if } j = 0, \\ \mathbb{Z}^{2g+1} \oplus \mathbb{Z}_j & \text{if } j \neq 0, \end{cases} \\
&K^1(X^3, H = 0) = H^1(X^3, \mathbb{Z}) \oplus H^3(X^3, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{2g+2} & \text{if } j = 0, \\ \mathbb{Z}^{2g+1} & \text{if } j \neq 0, \end{cases} \\
&K^0(X^3, H = k) = H^2(X^3, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{2g+1} & \text{if } j = 0, \\ \mathbb{Z}^{2g} \oplus \mathbb{Z}_j & \text{if } j \neq 0, \end{cases} \\
&K^1(X^3, H = k) = H^1(X^3, \mathbb{Z}) \oplus H^3(X^3, \mathbb{Z})/kH^3(X^3, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{2g+1} \oplus \mathbb{Z}_k & \text{if } j = 0, \\ \mathbb{Z}^{2g} \oplus \mathbb{Z}_k & \text{if } j \neq 0. \end{cases}
\end{aligned} \tag{5.9}$$

In fact T-duality is the interchange of j and k. Results in the twisted K-groups $K^0(X^3, H)$ and $K^1(X^3, H)$ being interchanged, which corresponds to the fact that RR field strengths are classified by $K^0(X^3, H)$ in type IIA string theory and by $K^1(X^3, H)$ in IIB. This means that applying the isomorphism between the two K-groups one can find the new RR field strengths from the old ones. Indeed one simply interchanges the $\mathbb{Z}^{2g}$ between H^1 and H^2 and the rest of the cohomology groups are swapped $H^0 \leftrightarrow H^1$, $H^2 \leftrightarrow H^3$.

Lens spaces. Let us consider the case of a linking two-sphere. It gives an isomorphism of the twisted K-theories of Lens spaces $L(1, p) = S^3/\mathbb{Z}_p$ (which is the Eilenberg–MacLane space, see remark 6.1) [22]:

$$K^\ell(L(1, j), H = k) \cong K^{\ell+1}(L(1, k), H = j). \tag{5.10}$$

Recall that $L(1, p) = \mathbb{S}^3/\mathbb{Z}_p$ is the total space of the circle bundle over the two-sphere with Chern class equal to p times the generator of $H^2(\mathbb{S}^2, \mathbb{Z}) \cong \mathbb{Z}$. In particular, $L(1, 1) = \mathbb{S}^3$ and $L(1, 0) = \mathbb{S}^2 \times S^1$. The Lens space $L(2, j)$ can be considered as the nonsingular quotient $X = \mathbb{S}^5/\mathbb{Z}_j$ when $j \neq 0$ and $X = \mathbb{C}P^2 \times S^1$ when $j = 0$. Integral cohomology groups of the oriented Lens space $L(n, q)$ are [47]

$$\begin{aligned}
&H^*(L(n, q), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{in dimension } 0, \\ \mathbb{Z}_q & \text{in dimensions } 2n, \\ 0 & \text{otherwise,} \end{cases} \qquad H^{0\leqslant p\leqslant 5}(\mathbb{C}P^2 \times S^1) = \mathbb{Z}, \\
&H^0(L(2, j), \mathbb{Z}) = H^5(L(2, j), \mathbb{Z}) = \mathbb{Z}, \quad H^2(L(2, j), \mathbb{Z}) = H^4(L(2, j), \mathbb{Z}) = \mathbb{Z}_j.
\end{aligned} \tag{5.11}$$

[8] Example [15]: in the flat case and for the Euclidean group $G = \mathbb{R}^{2n} \rtimes \mathbf{SO}(2n)$, $K = \mathbf{SO}(2n)$ and $\Gamma \subset G$ is a Bieberbach group, i.e. Γ is a uniform lattice in G. Also a generalization of *noncommutative flat manifolds* can be defined by regarding $C^*(\Gamma, \sigma)$ as such an object. In addition, σ is any group two-cocycle on Γ, due to the fact that $K_*(C^*(\Gamma, \sigma)) \cong K^*(\Gamma\backslash G/K)$.

A nontrivial quotient of string solution $AdS^5 \times \mathbb{S}^5$ can be associated with circle bundles on $\mathbb{C}P^2$ (see for detail [22]). It is easy to see that H-flux is only possible for the trivial bundle $j = 0$, as the nontrivial bundles have trivial third cohomology. T-duality relates the trivial bundle with $H = j$ to the bundle with the first Chern class j and no flux [22].

Remark 5.2. The case $j = 1$ of the T-duality mentioned above has been studied in [48], and was observed that the spacetime on the IIA side is not $Spin$-manifold, making the duality quite nontrivial. T-dualities considered in this section are interesting because IIB string theory on $AdS^5 \times \mathbb{S}^5$ is well understood. Note that the resulting RR fluxes are easily computed. Indeed, following the lines of [22] we can start with N units of G_5-flux supported on $L(2, j)$ in IIB. Then in IIA theory there will be N units of G_4-flux supported on $\mathbb{C}P^2$ and j units of H-flux supported on $H^2(\mathbb{C}P^2, \mathbb{Z}) \otimes H^1(S^1, \mathbb{Z})$.

5.1. K-groups for branes and fluxes on $\mathbb{R}P^7 \times \mathbb{T}^3$

Remark 5.3. Note that factors $\mathbb{Z}^{2g}$ do not play important role in what follows. In fact, we can ignore them and consider the two-sphere $\Sigma_{g=0} = \mathbb{S}^2$ or two-torus $\Sigma_{g=1} = \mathbb{T}^2$ cases.

Proposition 5.2. *Let M be a contractible. For freely acting of a group G on M*, $H_n(G, A) = H_n(X, A)$, *and* $H^n(G, A) = H^n(X, A)$, *where A is a trivial G-bundle and* $X \equiv G\backslash M$ *is the orbit space G in M, providing factor topology and canonic mapping* π *from M to X.*

Let as before G be a connected Lie group, K be a maximal compact subgroup and Γ be a closed discrete subgroup without torsion. (The Γ-acting on G/K is defined by $\gamma(gK) = (\gamma g)K$.) In accordance with proposition 5.2, $H_n(\Gamma, A) = H_n(\Gamma\backslash G/K, A)$, $H^n(\Gamma, A) = H^n(\Gamma\backslash G/K, A)$, where A is a trivial Γ-module. As an example, let $G = \mathbb{R}^n$, $\Gamma = \mathbb{Z}^n$ and $K = \{0\}$. Then $X = \mathbb{T}^n$, and

$$H^p(\mathbb{T}^n, \mathbb{Z}) = H_p(\mathbb{T}^n, \mathbb{Z}) = \mathbb{Z}^{\binom{n}{p}}. \tag{5.12}$$

The nontrivial classes for $\mathbb{T}^3$ are

$$\begin{aligned}
&H^0(\mathbb{T}^3, \mathbb{Z}) = H_0(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z}, && K^0(\mathbb{T}^3) = H^0(\mathbb{T}^3, \mathbb{Z}) \oplus H^2(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^3,\\
&H^1(\mathbb{T}^3, \mathbb{Z}) = H_1(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z}^3, && K^1(\mathbb{T}^3) = H^1(\mathbb{T}^3, \mathbb{Z}) \oplus H^3(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^3,\\
&H^2(\mathbb{T}^3, \mathbb{Z}) = H_2(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z}^3, && K_0(\mathbb{T}^3) = H_0(\mathbb{T}^3, \mathbb{Z}) \oplus H_2(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^3,\\
&H^3(\mathbb{T}^3, \mathbb{Z}) = H_3(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z}, && K_1(\mathbb{T}^3) = H_1(\mathbb{T}^3, \mathbb{Z}) \oplus H_3(\mathbb{T}^3, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^3.
\end{aligned} \tag{5.13}$$

The torsion and (co)homology of $\mathbb{R}P^n$ are

$$\begin{gathered}
H^n(\mathbb{R}P^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n \text{ odd} \\ \mathbb{Z}_2, & \text{if } n \text{ even} \end{cases} \quad H_n(\mathbb{R}P^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases}\\
H^k(\mathbb{R}P^n, \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, \ldots, H_k(\mathbb{R}P^n, \mathbb{Z}) = \mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \ldots, k = 0, \ldots, n.
\end{gathered} \tag{5.14}$$

Therefore, torsion, (co)homology and lower K-groups of $\mathbb{R}P^7$ become

$$\begin{aligned}
&H_0(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}, && H^0(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}, && \\
&H_1(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}_2, && H^1(\mathbb{R}P^7, \mathbb{Z}) = 0, && \\
&H_2(\mathbb{R}P^7, \mathbb{Z}) = 0, && H^2(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}_2, && K_0(\mathbb{R}P^7) = \mathbb{Z},\\
&H_3(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}_2, && H^3(\mathbb{R}P^7, \mathbb{Z}) = 0, && K_1(\mathbb{R}P^7) = \mathbb{Z} \oplus \mathbb{Z}_8,\\
&H_4(\mathbb{R}P^7, \mathbb{Z}) = 0, && H^4(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}_2, && K^0(\mathbb{R}P^7) = \mathbb{Z} \oplus \mathbb{Z}_8,\\
&H_5(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}_2, && H^5(\mathbb{R}P^7, \mathbb{Z}) = 0, && K^1(\mathbb{R}P^7) = \mathbb{Z},\\
&H_6(\mathbb{R}P^7, \mathbb{Z}) = 0, && H^6(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}_2, && \\
&H_7(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}, && H^7(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}. &&
\end{aligned} \tag{5.15}$$

The K-theories of $\mathbb{R}P^7 \times \mathbb{T}^3$ can be combined in order to find the K-theory of the product using the Künneth formula

$$H_n(M_1 \times M_2) = \bigoplus_j (H_j(M_1) \otimes H_{n-j}(M_2)) \bigoplus \Big(\bigoplus_j \mathrm{Tor}(H_j(M_1), H_{n-j-1}(M_2)) \Big). \quad (5.16)$$

Remark 5.4. Let M be a topological space (or finite CW-complex). Because of the Künneth formula one has $K_\ell(\mathbb{T}^n) \cong K_\ell(S^1)^{\oplus 2^{n-1}} \cong \mathbb{Z}^{\oplus 2^{n-1}}$ for $\ell = 0, 1$. Thus, the following isomorphisms are valid:

$$\begin{aligned} &K_0(M \times \mathbb{T}^n) \cong \big(\widetilde{K}_0(M) \oplus K_1(M) \oplus \mathbb{Z}\big)^{\oplus 2^{n-1}}, \\ &K_1(M \times \mathbb{T}^n) \cong \big(K_1(M) \oplus \widetilde{K}_0(M) \oplus \mathbb{Z}\big)^{\oplus 2^{n-1}} \Longrightarrow \\ &K_0(M \times \mathbb{T}^n) \cong K_1(M \times \mathbb{T}^n). \end{aligned} \quad (5.17)$$

Isomorphism (5.17) is in fact the T-duality and describes a relationship between type IIB and type IIA D-branes on the spacetime $M \times \mathbb{T}^n$. This isomorphism exchanges wrapped D-branes with unwrapped D-branes. In addition, the powers of 2^{n-1} give the expected multiplicity of Dp-brane charges arising from wrapping all higher stable D-branes on various cycles of the torus $\mathbb{T}^n$.

Remark 5.5. For the compactification of type II (or type I) on an n-torus $\mathbb{T}^n$, one can get the following result [49]:[9]

$$\begin{aligned} &[K(KO)](M \times \mathbb{T}^n, \mathbb{T}^n) = \bigoplus_{k=0}^{n} [\widetilde{K}(\widetilde{KO})]^{-k}(M)^{\oplus \binom{n}{k}}, \\ &K(M \times \mathbb{T}^n, \mathbb{T}^n) \cong K^{-1}(M \times \mathbb{T}^n, \mathbb{T}^n). \end{aligned} \quad (5.18)$$

If $n = 1$, then under isomorphism (5.18) K-groups related to type II, $\widetilde{K}(M) \otimes_{\mathbb{Z}} K(S^1)$ maps to $\widetilde{K}(M) \otimes_{\mathbb{Z}} K^{-1}(S^1)$ with the summands $K(S^1)$ and $K^{-1}(S^1)$ interchanged. Thus, T-duality exchanges wrapped and unwrapped D-brane configurations. If $n > 1$, then (5.18) gives the anticipated degeneracies 2^{n-1} of brane charges arising from the higher supersymmetric branes wrapped on various cycles of the torus $\mathbb{T}^n$. One can also write down the explicit T-duality isomorphism between D-brane charges of type I compactified on a torus and those of the corresponding type II orientifold compactification [49].

The result for $\mathbb{R}P^7 \times \mathbb{T}^3$ is just the tensor product of the original K-theories because the K-theory of the torus has no torsion and so the Tor term in the Künneth formula is trivial:

$$\begin{aligned} &K_0(\mathbb{R}P^7 \times \mathbb{T}^3) = ((K_0(\mathbb{R}P^7) \otimes K_0(\mathbb{T}^3)) \oplus (K_1(\mathbb{R}P^7) \otimes K_1(\mathbb{T}^3)) = \mathbb{Z}^2 \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_8^2, \\ &K_1(\mathbb{R}P^7 \times \mathbb{T}^3) = ((K_0(\mathbb{R}P^7) \otimes K_1(\mathbb{T}^3)) \oplus (K_1(\mathbb{R}P^7) \otimes K_0(\mathbb{T}^3)) = \mathbb{Z}^2 \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_8^2, \\ &K^0(\mathbb{R}P^7 \times \mathbb{T}^3) = ((K^0(\mathbb{R}P^7) \otimes K^0(\mathbb{T}^3)) \oplus (K^1(\mathbb{R}P^7) \otimes K^1(\mathbb{T}^3)) = \mathbb{Z}^2 \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_8^2, \\ &K^1(\mathbb{R}P^7 \times \mathbb{T}^3) = ((K^0(\mathbb{R}P^7) \otimes K^1(\mathbb{T}^3)) \oplus (K^1(\mathbb{R}P^7) \otimes K^0(\mathbb{T}^3)) = \mathbb{Z}^2 \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_8^2. \end{aligned} \quad (5.19)$$

The universal coefficient theorem yields to the K-cohomology groups

$$K^0(\mathbb{R}P^7 \times \mathbb{T}^3) = K^1(\mathbb{R}P^7 \times \mathbb{T}^3) = \mathbb{Z}^2 \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_8^2. \quad (5.20)$$

The untwisted K-theories associated with real projective spaces $\mathbb{R}P^{2k+1}$ can be obtained identically. It is known that in type IIA string theory D-branes are classified by a group K_1 while RR field strengths are classified by a group K^0. In the case of $\mathbb{R}P^7 \times \mathbb{T}^3$, both

[9] We use notation KO for group $K_{\mathbb{R}}$.

groups are $\mathbb{Z}^2 \oplus \mathbb{Z}^4 \oplus \mathbb{Z}_8^2$. Therefore, D-branes are classified by $K_1(\mathbb{R}P^7 \times \mathbb{T}^3)$ and the physical interpretation of the generators and branes wraps the $\mathbb{T}^3$ and the $\mathbb{R}P^7$ are similar to the description given in [24].

5.2. *T-dualizing on* $\mathbb{C}P^3 \times \Sigma_g \times \mathbb{T}^2$

5.2.1. Untwisted K-theory for the case $M = \mathbb{C}P^3 \times \Sigma_g \times \mathbb{T}^2$. Here, we dualize the $\mathbb{R}P^7 \times X^3$ case with no H-flux. Let the manifold X^3 admit free cycle actions. In particular, both manifolds X^3 and $\mathbb{R}P^7$ are circle bundles over space the Riemannian two-space Σ_g and the complex projective spaces $\mathbb{C}P^3$, respectively,

$$\begin{array}{ccccc} S^1 & \longrightarrow & X^3 \; \widehat{S}^1 & \longrightarrow & \mathbb{R}P^7 \\ & & \pi\big\downarrow & & \pi\big\downarrow \\ & & \Sigma_g & & \mathbb{C}P^3. \end{array} \tag{5.21}$$

The nontrivial cohomology classes of the base spaces are

$$H^0(\mathbb{C}P^3, \mathbb{Z}) = H^2(\mathbb{C}P^3, \mathbb{Z}) = H^4(\mathbb{C}P^3, \mathbb{Z}) = H^6(\mathbb{C}P^3, \mathbb{Z}) = \mathbb{Z}. \tag{5.22}$$

We can T-dualize both circle fibres following the lines of [22, 24]. Note that the T-duality exchanges the integrals of the H-fluxes over the circle fibres with the Chern classes. Since the original H-flux vanishes the dual Chern classes also vanish. Therefore, the dual spacetime $\widehat{M}$ consists of the product of two trivial circle bundles over the original base

$$\widehat{M} = \mathbb{C}P^3 \times \Sigma_g \times S^1_\alpha \times S^1_\beta, \tag{5.23}$$

where α and β are associated with the Chern classes $c_1 \in H^2(\Sigma_g, \mathbb{Z})$ and $c_1 \in H^2(\mathbb{C}P^3, \mathbb{Z})$ of the bundles over Σ_g and $\mathbb{C}P^3$, respectively. The dual H-flux, $\widehat{H}$, is the sum of two H-fluxes that integrate to the Chern classes.

5.2.2. Twisted K-theory for the case $\widehat{M} = \mathbb{C}P^3 \times \Sigma_g \times S^1_\alpha \times S^1_\beta$. The twisted K-theory of $\widehat{M}$, as the theory which has been performed due to an even number of T-dualities, must agree with the untwisted K-theory of the space $\mathbb{R}P^7 \times X^3$. The Künneth formula for K-homology is

$$0 \longrightarrow K_*(M_1) \otimes K_*(M_2) \longrightarrow K_*(M_1 \times M_2) \longrightarrow \mathrm{Tor}(K_*(M_1), K_*(M_2)) \longrightarrow 0. \tag{5.24}$$

The homology of X^3 with coefficients in $\mathbb{Z}$ can be computed using the result (5.9) and the universal coefficient theorem.

Theorem 5.1 (universal coefficient theorem)**.** *For any space X and the associated Abelian group G the following result holds:*

$\mathscr{T}$1. The homology group of X with coefficients in G has a splitting

$$H_p(X; G) \simeq H_p(X) \otimes G \oplus \mathrm{Tor}(H_{p-1}(X, G)).$$

$\mathscr{T}$2. The cohomology group of X with coefficients in G has a splitting

$$H^p(X; G) \simeq \mathrm{Hom}(H_p(X), G) \oplus \mathrm{Ext}(H_{p-1}(X), G).$$

Remark 5.6. The (splittings) isomorphisms given by the universal coefficient theorem are said to be unnatural isomorphisms. The following maps of exact sequences are natural:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_p(X) \otimes G & \longrightarrow & H_p(X; G) & \longrightarrow & \mathrm{Tor}(H_{p-1}(X), G) & \longrightarrow & 0 \\ 0 & \longrightarrow & H^p(X, \mathbb{Z}) \otimes G & \longrightarrow & H^p(X; G) & \longrightarrow & \mathrm{Tor}(H^{p+1}(X; \mathbb{Z}), G) & \longrightarrow & 0 \\ 0 & \longleftarrow & \mathrm{Hom}(H_p(X), G) & \longleftarrow & H^p(X; G) & \longleftarrow & \mathrm{Ext}(H_{p+1}(X), G) & \longleftarrow & 0. \end{array}$$

For the integer cohomology ($G = \mathbb{Z}$), we have the following result.

Corollary 5.1. *For any space X for which $H_\ell(X)$ and $H_{\ell-1}(X)$ are finite generated $\mathbb{Z}$-modules, it follows*

$$H^\ell(X) \simeq \mathscr{F}_\ell(X) \oplus \mathrm{Tor}(H_{\ell-1}(X)). \tag{5.25}$$

Here, $\mathscr{F}_\ell(X)$ and $\mathrm{Tor}(H_{\ell-1}(X))$ are the free and torsion parts of $H_\ell(X)$ and $H_{\ell-1}(X)$, respectively.

We can evaluate the homology of the product using the Künneth formula. It gives

$$\begin{aligned} K_0(\mathbb{R}P^7\times X^3) &= (K_0(\mathbb{R}P^7)\otimes K_0(X^3))\oplus(K_1(\mathbb{R}P^7)\otimes K_1(X^3))\\ &\quad\oplus \mathrm{Tor}(K_0(\mathbb{R}P^7), K_1(X^3))\oplus\mathrm{Tor}(K_1(\mathbb{R}P^7), K_0(X^3)),\\ K_1(\mathbb{R}P^7\times X^3) &= (K_0(\mathbb{R}P^7)\otimes K_1(X^3))\oplus(K_1(\mathbb{R}P^7)\otimes K_0(X^3))\\ &\quad\oplus \mathrm{Tor}(K_0(\mathbb{R}P^7), K_0(X^3))\oplus\mathrm{Tor}(K_1(\mathbb{R}P^7), K_1(X^3)). \end{aligned} \tag{5.26}$$

Remark 5.7. Note that $\mathrm{Tor}(A, B)$ vanishes unless both A and B contain torsion components. If m and n are positive integers, then $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{(m,n)}$, where (m, n) denotes the greatest common divisor of m and n.

Therefore, we get the following result:

$$\begin{aligned} &K_0(\mathbb{R}P^7\times X^3; H=0, j=0) = (\mathbb{Z}^{2g+2}\otimes\mathbb{Z})\oplus(\mathbb{Z}^{2g+2}\otimes(\mathbb{Z}\oplus\mathbb{Z}_8))\oplus 0\oplus 0 = \mathbb{Z}^{2g+3}\oplus\mathbb{Z}_8,\\ &K_0(\mathbb{R}P^7\times X^3; H=0, j\neq 0) = ((\mathbb{Z}^{2g+1}\oplus\mathbb{Z}_j)\otimes\mathbb{Z})\oplus(\mathbb{Z}^{2g+1}\otimes(\mathbb{Z}\oplus\mathbb{Z}_8))\\ &\qquad\oplus\mathrm{Tor}(\mathbb{Z}_j,\mathbb{Z}_8)\oplus 0 = \mathbb{Z}^{2g+2}\oplus\mathbb{Z}_j\oplus\mathbb{Z}_8\oplus\mathbb{Z}_{(j,8)},\\ &K_0(\mathbb{R}P^7\times X^3; H=k, j=0) = (\mathbb{Z}^{2g+1}\otimes\mathbb{Z})\oplus((\mathbb{Z}^{2g+1}\oplus\mathbb{Z}_k)\otimes(\mathbb{Z}\oplus\mathbb{Z}_8))\oplus 0\oplus 0\\ &\qquad= \mathbb{Z}^{2g+2}\oplus\mathbb{Z}_k\oplus\mathbb{Z}_8\oplus\mathbb{Z}_{(k,8)},\\ &K_0(\mathbb{R}P^7\times X^3; H=k, j\neq 0) = ((\mathbb{Z}^{2g}\oplus\mathbb{Z}_j)\otimes\mathbb{Z})\oplus((\mathbb{Z}^{2g}\oplus\mathbb{Z}_k)\otimes(\mathbb{Z}\oplus\mathbb{Z}_8))\\ &\qquad\oplus\mathrm{Tor}(\mathbb{Z}_j,\mathbb{Z}_8)\oplus 0 = \mathbb{Z}^{2g+1}\oplus\mathbb{Z}_j\oplus\mathbb{Z}_k\oplus\mathbb{Z}_8\oplus\mathbb{Z}_{(j,8)}\oplus\mathbb{Z}_{(k,8)},\\ &K_1(\mathbb{R}P^7\times X^3; H=0, j=0) = (\mathbb{Z}^{2g+2}\otimes(\mathbb{Z}\oplus\mathbb{Z}_8))\oplus(\mathbb{Z}^{2g+2}\otimes\mathbb{Z})\oplus 0\oplus 0 = \mathbb{Z}^{2g+3}\oplus\mathbb{Z}_8,\\ &K_1(\mathbb{R}P^7\times X^3; H=0, j\neq 0) = ((\mathbb{Z}^{2g+1}\oplus\mathbb{Z}_j)\otimes(\mathbb{Z}\oplus\mathbb{Z}_8))\oplus(\mathbb{Z}^{2g+1}\otimes\mathbb{Z})\oplus 0\oplus 0\\ &\qquad= \mathbb{Z}^{2g+2}\oplus\mathbb{Z}_j\oplus\mathbb{Z}_8\oplus\mathbb{Z}_{(j,8)},\\ &K_1(\mathbb{R}P^7\times X^3; H=k, j=0) = (\mathbb{Z}^{2g+1}\otimes(\mathbb{Z}\oplus\mathbb{Z}_8))\oplus((\mathbb{Z}^{2g+1}\oplus\mathbb{Z}_k)\otimes\mathbb{Z})\oplus 0\\ &\qquad\oplus\mathrm{Tor}(\mathbb{Z}_k,\mathbb{Z}_8) = \mathbb{Z}^{2g+2}\oplus\mathbb{Z}_k\oplus\mathbb{Z}_8\oplus\mathbb{Z}_{(k,8)},\\ &K_1(\mathbb{R}P^7\times X^3; H=k, j\neq 0) = ((\mathbb{Z}^{2g}\oplus\mathbb{Z}_j)\otimes(\mathbb{Z}\oplus\mathbb{Z}_8))\oplus((\mathbb{Z}^{2g}\oplus\mathbb{Z}_k)\otimes\mathbb{Z})\oplus 0\\ &\qquad\oplus\mathrm{Tor}(\mathbb{Z}_k,\mathbb{Z}_8) = \mathbb{Z}^{2g+1}\oplus\mathbb{Z}_j\oplus\mathbb{Z}_k\oplus\mathbb{Z}_8\oplus\mathbb{Z}_{(j,8)}\oplus\mathbb{Z}_{(k,8)}. \end{aligned} \tag{5.27}$$

In the absence of the Neveu–Schwarz (NS) flux, K^1 group describes RR field strengths in IIB string theory, which we write locally as $G_p = dG_{p-1}$. It has been suggested that RR field strengths are classified by twisted K-theory. Field strengths satisfy $d_3G_p = (Sq^3 + H\cup)G_p = 0$. Here, Sq^3 is the Steenrod squares that takes torsion p-classes to torsion $(p + 3)$-classes[10]. In the case of S-duality, a configuration in which the G_3-flux valued in $\mathbb{Z} \subset H^3(\mathbb{R}P^7 \times X^3)$ is nonzero, we can find another configuration which corresponds to a class in the K-theory twisted by the original G_3. Let us consider the dualizing of $\mathbb{R}P^7 \times \mathbb{T}^3$ with no H-flux, example

[10] In the classical limit, type II supergravity, we can forget the flux quantization condition and look at real cohomology. In fact, we can no longer see the Sq^3 term. This theory contains RR potentials C_{p-1}, a NS three-form H and a gauge-invariance $C_{p-1} \to C_{p-1} + d\Lambda_{p-2} + H \wedge \Lambda_{p-4}$, for any set of forms Λ_k. It follows that there are two natural field strengths $G_p = dC_{p-1}$ and $F_p = dC_{p-1} + H \wedge C_{p-3}$. In addition, G_p is closed and F_p is gauge-invariant.

from section 5.1. In this case, we are looking for a trivial G-bundle (see proposition 5.2), $\Sigma_{g=1} = \mathbb{T}^2$ and $X^3 = \mathbb{T}^3$. Then, T-dualizing the circles becomes

$$\begin{array}{llll} \widehat{S}^1 \longrightarrow \mathbb{R}P^7 & S^1 \times \mathbb{T}^2 & & \\ \quad\quad\ \ \pi\downarrow & \ \ \pi\downarrow & & \\ \quad\quad\ \mathbb{C}P^3 \times \mathbb{T}^3 \times \mathbb{T}^2 & \xleftrightarrow{\text{T-Duality}} & \underbrace{\mathbb{C}P^3 \times \Sigma_g \times S^1_\alpha \times S^1_\beta}_{H\neq 0} \xleftrightarrow{\text{2T-Dualities}} & \underbrace{\mathbb{R}P^7 \times \mathbb{T}^3}_{H=0}. \end{array} \tag{5.28}$$

The twisted K-theory associated with space $\widehat{M} = \mathbb{C}P^3 \times \Sigma_g \times S^1_\alpha \times S^1_\beta$, as we have performed an even number of T-dualities, must agree with the untwisted K-theory of the original space $\mathbb{R}P^7 \times X^3$:

$$\begin{array}{llll} \widehat{S}^1 \longrightarrow \mathbb{R}P^7 & S^1 \longrightarrow X^3 & & \\ \quad\quad\ \ \pi\downarrow & \quad\quad\ \pi\downarrow & & \\ \quad\quad\ \mathbb{C}P^3 \quad \times & \Sigma_g \times \mathbb{T}^2 & \xleftrightarrow{\text{T-duality}} \underbrace{\mathbb{C}P^3 \times \Sigma_g \times S^1_\alpha \times S^1_\beta}_{H\neq 0} \xleftrightarrow{\text{2T-dualities}} & \underbrace{\mathbb{R}P^7 \times X^3}_{H=0}. \end{array} \tag{5.29}$$

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Appendix

A.1. Eilenberg–MacLane spaces

Definition 6.1. *Let Γ be a discrete group. A based topological space (or CW complex) X is called an Eilenberg–MacLane space of type $K(\Gamma, n)$, where $n \geqslant 1$, if $\Gamma_k(X) = \Gamma$ for $k = n$ and $\Gamma_k(X) = 0$ for $k \neq n$. It means that all the homotopy groups $\Gamma_k(X)$ are trivial except for $\Gamma_n(X)$, which is isomorphic to Γ.*

Note that $K(\Gamma, 0)$ is a CW complex with $\Gamma_0 = \Gamma$ having contractible components. For $n = 1$ and finitely generated group Γ the spaces $K(\Gamma, 1)$ are well known: $K(\mathbb{Z}, 1) = S^1$, $K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$ and $K(\mathbb{Z}_m, 1) = L^\infty_m$ for $m > 2$. For $n \geqslant 2$ group, Γ must be Abelian. For any Abelian group Γ with $n \geqslant 2$, there exists an Eilenberg–MacLane space of type $K(\Gamma, n)$ which can be constructed as a CW complex[11]. In general, they are more complicated objects which play a fundamental role in the connection between homotopy and (co)homology.

Remark 6.1. Recall that the simplest examples of Eilenberg–MacLane spaces are

$\mathscr{R}1$. $K(\mathbb{Z}, 2) \sim \mathbb{C}P^\infty = S^\infty/S^1$; $H^*(\mathbb{C}P^\infty; \mathbb{Z}) \simeq \mathbb{Z}_p, t \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ for all primes $p \geqslant 2$.

[11] The Whitehead theorem implies that there is a unique $K(\Gamma, n)$ space up to homotopy equivalence in the category of topological spaces of the homotopy type of a CW complex.

$\mathscr{R}2$. $K(\mathbb{Z}_{p^n}, 1) \sim S^\infty/\mathbb{Z}_{p^n} = \lim_{N\to\infty} L^{2N+1}_{p^n}(1, 1, \ldots, 1)$. The general lens space $L^{2n-1}_m(q_1, \ldots, q_{n-1})$ of dimension $2n - 1$, is defined as the orbit space of the sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ under the action of the group $\mathbb{Z}_m$ given by $(z^1, \ldots, z^n) \mapsto (\mathrm{e}^{2\pi\sqrt{-1}/m}z^1, \mathrm{e}^{2\pi\sqrt{-1}q_1/m}z^2, \ldots, \mathrm{e}^{2\pi\sqrt{-1}q_{n-1}/m}z^n)$, where each q_ℓ is relatively prime to m. With this action of $\mathbb{Z}_m$, we set $L^{2n-1}_m(q_1, \ldots, q_{n-1}) = \mathbb{S}^{2n-1}/\mathbb{Z}_m$. In particular, $K(\mathbb{Z}_2, 1) = \mathbb{R}P^2 = \lim_{N\to\infty}\mathbb{R}P^N$.

$\mathscr{R}3$. $\Gamma = F$ (a free group), $K(F, 1) \sim S^1 \vee \cdots \vee S^1$ (a bouquet of circles).

Any two Eilenberg–MacLane spaces of $K(\Gamma, n)$ are weakly homotopy equivalent. Also $K(\Gamma, n)$ is a homotopy commutative H-space. The following result holds:

$$K(\Gamma_1 \times \Gamma_2, n) = K(\Gamma_1, n) \times K(\Gamma_2, n), \qquad \Omega(K(\Gamma, n)) = K(\Gamma, n - 1), \tag{A.1}$$

where $\Omega(K(\Gamma, n))$ is the loop space relative to some base point. The second equation is essentially a consequence of the homotopy groups isomorphism $\pi_\ell(\Omega(X)) \simeq \pi_{\ell+1}(X)$.

Theorem 6.1 (Hurewicz isomorphism theorem)**.** *For n-connected cell complex C the groups* $\pi_{n+1}(C)$ *and* $H_{n+1}(C; \mathbb{Z})$ $(n > 0)$ *are isomorphic.*

By Hurwitz's theorem 6.1, $H_n(K(\Gamma, n); \mathbb{Z}) \simeq \pi_n(K(\Gamma, n)) \simeq \Gamma$. It follows that there is a canonical isomorphism $H^n(K(\Gamma, n); G) \simeq \mathrm{Hom}(\Gamma, G)$, where G is any Abelian group and $\mathrm{Hom}(\Gamma, G)$ denotes the additive group of homomorphisms from the Abelian group Γ to G.

Partial computing of $H^*(K(\Gamma, n), \mathbb{Q})$ *ring*. Let Γ be a finitely generated Abelian group and let $\Gamma = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus G$, where G is a finite group. Then, respectively, $K(\Gamma, n) = K(\mathbb{Z}, n) \times \cdots \times K(\Gamma, n) \times K(G, n)$. Because of the Künneth formulae, we get

$$H^*(K(\Gamma, n); \mathbb{Q}) = H^*(K(\mathbb{Z}, n); \mathbb{Q}) \otimes \cdots \otimes H^*(K(\mathbb{Z}, n); \mathbb{Q}) \otimes H^*(K(G, n); \mathbb{Q}). \tag{A.2}$$

Note that if integer cohomologies of a topological space are finite then its rational cohomologies are trivial; $H^*(K(G, n); \mathbb{Q}) = H^*(pt; \mathbb{Q})$. Therefore, we have to compute $H^*(K(\mathbb{Z}, n); \mathbb{Q})$, neglecting the last term in equation (A.2).

Let $\mathbf{k}$ be a field, $\Lambda_{\mathbf{k}}(x_1, \ldots, x_n)$ $(\dim x = n)$ external algebra generated by $x_1, \ldots, x_n$, i.e. $\mathbf{k}$-algebra with generators x_ℓ and relations $x_k x_\ell = -x_\ell x_k$, $x_\ell^2 = 0$. The dimension of this algebra is 2^m and its basis form monoms $x_{\ell_1} \ldots x_{\ell_s}$, $1 \leqslant \ell_1 < \cdots < \ell_s \leqslant m$. Finally, $\mathbf{k}[x_1, \ldots, x_m]$ denotes an algebra of polynomials with coefficients in $\mathbf{k}$.

Theorem 6.2.

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \begin{cases} \Lambda_{\mathbb{Q}}(x) & \text{for odd } n, \\ \mathbb{Q}[x] & \text{for even } n. \end{cases} \tag{A.3}$$

The formula $H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x)(\dim x = n)$ *has a simple meaning:* $H^*(K(\mathbb{Z}, n); \mathbb{Q}) = H^*(\mathbb{S}^n; \mathbb{Q})$*. The formula* $H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \mathbb{Q}[x](\dim x = n)$ *means that* $H^\ell(K(\mathbb{Z}, n); \mathbb{Q}) = \mathbb{Q}$ *for* $\ell = 0, n, 2n, \ldots$, $H^\ell(K(\mathbb{Z}, n); \mathbb{Q}) = 0$ *for other values of* ℓ *and the element* x^q, $0 \neq x \in H^n(K(\mathbb{Z}, n); \mathbb{Q})$*, generates* $H^{qn}(K(\mathbb{Z}, n); \mathbb{Q})$ *over* $\mathbf{k}$.

Corollary 6.1. *Let* $\mathrm{rank}\,\Gamma = r$*, then*

$$H^*(K(\Gamma, n); \mathbb{Q}) = \begin{cases} \Lambda_{\mathbb{Q}}(x_1, \ldots, x_r) & \text{for odd } n, \\ \mathbb{Q}[x_1, \ldots, x_r] & \text{for even } n. \end{cases} \tag{A.4}$$

Let us consider the Eilenberg–MacLane complexes $K(\Gamma, n)$ for $\Gamma = \mathbb{Z}_{p^n}$ and p prime. We are looking for all cohomological operations modulo an arbitrary prime p. Recall that cohomologies of the space $K(\mathbb{Z}_p, 1) = L_p^\infty$ are known: $H^q(K(\mathbb{Z}_p, 1); \mathbb{Z}) = \mathbb{Z}_p$ for even q, and $H^q(K(\mathbb{Z}_p, 1); \mathbb{Z}) = 0$ for odd q. For positive q there is a multiplicative isomorphism:

$$H^*(K(\mathbb{Z}_p, 1); \mathbb{Z}) \cong \mathbb{Z}_p[x], \qquad x \in H^2(K(\mathbb{Z}_p, 1); \mathbb{Z}). \tag{A.5}$$

Theorem 6.3 (Fomenko and Fuks [50])**.** *For* $0 < q \leqslant n + 4p - 3$*, the ring* $H^q(K(\mathbb{Z}, n); \mathbb{Z}_p)$ *is isomorphic to* $H^q(K(\mathbb{Z}_p, n - 1); \mathbb{Z})$.

For further interesting examples of computing the groups $H^*(K(\Gamma, n), \mathbb{Q})$ we refer the reader to the book [50].

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